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Best L_2 Local Approximation*

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In 1934, Walsh noted that the Taylor polynomial of degree n can be obtained by taking the limit as $\epsilon \to 0^+$ of the net of nth degree polynomials which best approximate f in the closed discs $|z| \leqslant \epsilon$. Later, this result was generalized to rational approximation. In a recent paper, Shisha and the first two authors generalized this idea to the idea of *best local approximation*. In this paper, using a different technique, we study this problem in the L_2 setting. Consequently, better results follow under weaker hypotheses.

1. INTRODUCTION

When studying spline or piecewise polynomial approximation, it becomes apparent that the behavior of the approximants on small intervals is of great importance. Very little is known in this area. In this paper, we study this problem in the setting of best L_2 local approximation.

Let M be a class of functions in $L_2[0, \delta]$ where $\delta > 0$. Suppose that for each $\epsilon \in (0, \delta]$, a function f in $L_2[0, \delta]$ has a best $L_2[0, \epsilon]$ approximant $p_\epsilon(f)$ from M. If as $\epsilon \to 0^+$, the net $\{p_\epsilon(f)\}$ converges (in some sense) to some function $p_0(f) \in M$, we say that $p_0(f)$ is a best L_2 local approximant of f (at 0).

In [2], this problem was introduced and studied when the supremum norm is used. In the present paper, we use a different method to study the L_2 case. Consequently, "better" results can be obtained under weaker conditions. More generally, we consider the set $P_0(f)$ of cluster points of the net $\{p_\epsilon(f)\}$ as $\epsilon \to 0^+$. Examples can be obtained to show that $P_0(f)$ may be the empty set, a singleton, or a linear convex set with more than one element; but it is not known, even in the "simplest" case if $P_0(f)$ is a convex set. Much is left unexplored due to the diverse nature of this problem.

In Section 2, we will prove the main theorems, including conditions to guarantee that $P_0(f)$ is a singleton. In the Section 3, we will give a very preliminary result on $P_0(f)$ and will show the connection between best L_2 local quasi-rational approximants and Padé approximants.

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2. Main Results

Let $u_1, \ldots, u_n \in C^{n-1}[0, \delta]$ for some positive δ such that the wronskian matrix

$$W_n(u_1, \ldots, u_n; x) = \begin{bmatrix} u_1(x) & \cdots u_n(x) \\ u_1'(x) & \cdots u_n'(x) \\ u_1^{(n-1)}(x) & \cdots u_n^{(n-1)}(x) \end{bmatrix}$$

satisfies the condition $\det W_n(u_1, \ldots, u_n; 0) \neq 0$. Set $W_n \equiv W_n(u_1, \ldots, u_n; 0)$ and let $S_n \equiv S_n(u_1, \ldots, u_n)$ be the vector space spanned by $u_1, \ldots, u_n$. Hence, S_n is an n-dimensional subspace of $L_2[0, \delta]$. For an $f \in L_2[0, \epsilon]$, $0 < \epsilon \leqslant \delta$, we denote by $p_\epsilon(f)$ the (unique) best $L_2[0, \epsilon]$ approximant of f from S_n, i.e., $\| p_\epsilon(f) - f \|_\epsilon = \inf\{\| p - f\|_\epsilon : p \in S_n\}$, where $\| \cdot \|_\epsilon = \langle \cdot, \cdot \rangle_\epsilon^{1/2}$ and $\langle f, g \rangle_\epsilon = \int_0^\epsilon fg$. In this section, we will establish the following results.

Theorem 2.1. *Let $f \in C^{l-1}[0, \delta]$ where $\delta > 0$ and $1 \leqslant l \leqslant n$. Then for each j, $0 \leqslant j \leqslant l-1$,*

$$\lim_{\epsilon \to 0^+} (p_\epsilon(f) - f)^{(j)}(0) = 0. \tag{2.1}$$

Now let $e_j = [0, \ldots, 0, 1, 0, \ldots,]$ with 1 at the jth entry and $[c_{j,1}, \ldots, c_{j,n}]^T = W_n^{-1} e_j^T$, where, as usual, the superscript T indicates the transpose of a matrix. From the above theorem, one can easily obtain the following

Theorem 2.2. *Let $f \in C^{n-1}[0, \delta]$. Then the net $\{p_\epsilon(f)\}$, $0 < \epsilon \leqslant \delta$, converges uniformly in some neighborhood of 0 to some $p_0(f) \in S_n$ as $\epsilon \to 0^+$. Furthermore,*

$$p_0(f) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_{j,k} f^{(j-1)}(0)\, u_k. \tag{2.2}$$

It is clear (cf. [3]) that if $\det W_n \neq 0$, then $\{u_1, \ldots, u_n\}$ satisfies the Haar condition on some interval $[0, \epsilon_0]$ where $0 < \epsilon_0 \leqslant \delta$. We will next show that if $\{u_1, \ldots, u_n\}$ satisfies the Haar condition, then the condition $\det W_n \neq 0$ is necessary for the existence of best L_2 local approximants $p_0(f)$ for every f.

Theorem 2.3. *Let $\{u_1, \ldots, u_n\}$ satisfy the Haar condition on $[0, \delta]$, $\delta > 0$. Then $\det W_n \neq 0$ if and only if the net $\{p_\epsilon(f)\}$, as $\epsilon \to 0^+$, converges (uniformly on some $[0, \epsilon_0]$, $\epsilon_0 > 0$) for every $f \in C^{n-1}[0, \delta]$.*

In order to prove the above theorems, we need several lemmas.

LEMMA 2.1. *Let $D_n(\epsilon)$ be the determinant of the Gramian matrix*

$$\begin{bmatrix} \langle 1, 1\rangle_\epsilon & \langle 1, t\rangle_\epsilon & \cdots & \langle 1, t^{n-1}\rangle_\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ \langle t^{n-1}, 1\rangle_\epsilon & \langle t^{n-1}, t\rangle_\epsilon & \cdots & \langle t^{n-1}, t^{n-1}\rangle_\epsilon \end{bmatrix}$$

Then $D_n(\epsilon) = C_n\epsilon^n$. where

$$C_n = \int_0^1 \cdots \int_0^1 y_2 y_3^2 \cdots y_n^{n-1} \prod_{1\leqslant i<j\leqslant n} (y_j - y_i)\, dy_1 \cdots dy_n \tag{2.3}$$

is a non-zero constant independent of ϵ.

Proof. Consider the function

$$f(x_1, \ldots, x_n) = \begin{vmatrix} x_1 & x_1^2/2 & \cdots & x_1^n/n \\ x_2^2/2 & x_2^3/3 & \cdots & x_2^{n+1}/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ x_n^n/n & x_n^{n+1}/(n+1) & \cdots & x_n^{2n-1}/(2n-1) \end{vmatrix}.$$

It follows that

$$f_{1,\ldots,n}(x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{n-1} & x_n^{n} & \cdots & x_n^{2n-2} \end{vmatrix}$$

$$= x_2 x_3^2 \cdots x_n^{n-1} \prod_{1\leqslant i<j\leqslant n} (x_j - x_i)$$

where the subscripts of f indicate partial derivatives. Hence,

$$D_n(\epsilon) = f(\epsilon, \ldots, \epsilon) = \int_0^\epsilon \cdots \int_0^\epsilon x_2 x_3^2 \cdots x_n^{n-1} \prod_{1\leqslant i<j\leqslant n} (x_j - x_i)\, dx_1 \cdots dx_n$$

$$= C_n \epsilon^{n^2}$$

as asserted. It is clear that $C_n \neq 0$.

LEMMA 2.2. *Let $A = (a_{ij})$ be the $n \times n$ matrix*

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n+1 \\ \cdots & \cdots & \cdots & \cdots \\ n & n+1 & \cdots & 2n-1 \end{bmatrix}$$

and let $(j_1, \ldots, j_n)$ be any permutation of $(1, 2, \ldots, n)$. Then $\sum_{i=1}^n a_{i,j_i} = n^2$.

The proof of this lemma is clear by observing that $a_{i,j_i} = j_i + i - 1$.

LEMMA 2.3. *Let $g \in C[0, \delta]$, $\delta > 0$, such that $g(t) = o(t^{l-1})$ as $t \to 0^+$, where $1 \leqslant l \leqslant n$ and let $D_k(g, \epsilon)$ be the determinant of the matrix*

$$\begin{bmatrix} \langle 1, 1\rangle_\epsilon & \cdots & \langle 1, t^{k-2}\rangle_\epsilon & \langle 1, g\rangle_\epsilon & \langle 1, t^k\rangle_\epsilon & \cdots & \langle 1, t^{n-1}\rangle_\epsilon \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle t^{n-1}, 1\rangle_\epsilon & \cdots & \langle t^{n-1}, t^{k-2}\rangle_\epsilon & \langle t^{n-1}, g\rangle_\epsilon & \langle t^{n-1}, t^k\rangle_\epsilon & \cdots & \langle t^{n-1}, t^{n-1}\rangle_\epsilon \end{bmatrix}$$

Then $D_k(g, \epsilon) = o(\epsilon^{n^2})$ as $\epsilon \to 0^+$, for each $k = 1,..., l$.

Proof. Let $\delta(\epsilon) = \max\{\int_0^\epsilon |g(t)|\, t^j\, dt/\epsilon^{k+j}: 0 \leqslant j \leqslant n-1\}$. For $1 \leqslant k \leqslant l$, it is clear from the hypothesis that $\delta(\epsilon)$ tends to 0 as $\epsilon \to 0^+$. Now expand the determinant and estimate term by term. Then by using Lemma 2.2, we have $| D_k(g, \epsilon)| \leqslant \delta(\epsilon)(n!)\, \epsilon^{n^2}$ for $1 \leqslant k \leqslant l$.

Next, we consider a change of basis. Let $[c_{j,1} ,..., c_{j,n}]^T = W_n^{-1} e_j^T$ be as defined previously and let

$$v_j = (j-1)! \sum_{i=1}^{n} c_{j,i} u_i , \tag{2.4}$$

$1 \leqslant j \leqslant n$. The following lemma is self-evident.

LEMMA 2.4. *For each $j = 1,..., n$, $v_j(t) = t^{j-1} + o(t^n)$ as $t \to 0^+$.*

By using the above lemma, we immediately have

LEMMA 2.5. *Let $f \in C^{l-1}[0, \delta]$, $\delta > 0$, where $1 \leqslant l \leqslant n$. Then*

$$f(t) = \sum_{i=1}^{l} f^{(i-1)}(0)\, v_i(t)/(i-1)! + o(t^{l-1})$$

as $t \to 0^+$.

Next, we have the following estimate of the Gramian determinant of the basis $\{v_1 ,..., v_n\}$. It follows from Lemmas 2.1 and 2.4 and the proof of Lemma 2.3.

LEMMA 2.6. *Let $D_n(v_1 ,..., v_n ; \epsilon)$ be the determinant of the Gramian matrix*

$$\begin{bmatrix} \langle v_1 , v_1\rangle_\epsilon & \cdots & \langle v_1 , v_n\rangle_\epsilon \\ \cdots & \cdots & \cdots \\ \langle v_n , v_1\rangle_\epsilon & \cdots & \langle v_n , v_n\rangle_\epsilon \end{bmatrix}.$$

Then $D_n(v_1 ,..., v_n ; \epsilon) = (C_n + o(1))\, \epsilon^{n^2}$ as $\epsilon \to 0^+$, where C_n is the nonzero constant given in (2.3).

With the above preliminaries, we can now prove the main results. Clearly $\{v_1, \ldots, v_n\}$ is a basis of S_n. Let $f \in C^{l-1}[0, \delta]$ and write $p_\epsilon(f) = \sum_{k=1}^{n} \alpha_{k,\epsilon} v_k$. It is well-known that

$$\alpha_{k,\epsilon} = \frac{1}{D_n(v_1, \ldots, v_n; \epsilon)} \times \begin{vmatrix} \langle v_1, v_1\rangle_\epsilon & \cdots & \langle v_1, v_{k-1}\rangle_\epsilon & \langle v_1, f\rangle_\epsilon & \langle v_1, v_{k+1}\rangle_\epsilon & \cdots & \langle v_1, v_n\rangle_\epsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \langle v_n, v_1\rangle_\epsilon & \cdots & \langle v_n, v_{k-1}\rangle_\epsilon & \langle v_n, f\rangle_\epsilon & \langle v_n, v_{k+1}\rangle_\epsilon & \cdots & \langle v_n, v_n\rangle_\epsilon \end{vmatrix}$$

$$= \frac{1}{D_n(v_1, \ldots, v_n; \epsilon)} N_n(f, k; \epsilon).$$

From Lemma 2.5, we have

$$N_n(f, k; \epsilon) = \frac{f^{(k-1)}(0)}{(k-1)!} D_n(v_1, \ldots, v_n; \epsilon) + N_n(o(t^{l-1}), k; \epsilon).$$

By a proof similar to that of Lemma 2.3 and by using the estimates $v_j(t) = t^{j-1} + o(t^n)$, we have $N_n(o(t^{l-1}), k; \epsilon) = o(\epsilon^{n^2})$ as $\epsilon \to 0^+$ for $1 \leqslant k \leqslant l$. Hence, using Lemma 2.6, we have $\alpha_{k,\epsilon} = f^{(k-1)}(0)/(k-1)! + o(1)$ as $\epsilon \to 0^+$. That is, for $j = 0, \ldots, l-1$,

$$\begin{aligned}(p_\epsilon(f) - f)^{(j)}(0) &= \sum_{k=1}^{n} \alpha_{k,\epsilon} v_k^{(j)}(0) - f^{(j)}(0) \\ &= j!\alpha_{j+1,\epsilon} - f^{(j)}(0) \to 0\end{aligned}$$

as $\epsilon \to 0^+$. This completes the proof of Theorem 2.1. In particular, if $l = n$, then $p_\epsilon(f) = \sum_{k=1}^{n} \alpha_{k,\epsilon} v_k$ converges component-wise, and hence, converges uniformly in some neighborhood of 0. The limit function $p_0(f)$ is clearly given by (2.2) by using (2.4). Hence, we have Theorem 2.2 and one direction of Theorem 2.3. To complete the proof of Theorem 2.3, we note that $\langle p_\epsilon(f) - f, u\rangle_\epsilon = 0$ for every $u \in S_n$, and by standard arguments, since S_n is Haar, $p_\epsilon(f) - f$ has at least n sign changes on $[0, \epsilon]$. By applying Rolle's theorem and taking $\epsilon \to 0^+$, we have $(p_0(f) - f)^{(j)}(0) = 0$ for $j = 0, \ldots, n-1$. Hence, for all choices of $(f(0), \ldots, f^{(n-1)}(0))$, there always exist $(a_1, \ldots, a_n)$ such that, writing $p_0(f) = \sum_{k=1}^{n} a_k u_k$, we have $W_n[a_1, \ldots, a_n]^T = [f(0), \ldots, f^{(n-1)}(0)]^T$. That is, W_n is invertible.

It should be mentioned that the hypothesis on the system $\{u_1, \ldots, u_n\}$ can be slightly weakened. In particular, Theorems 2.1 and 2.3 still hold if if we assume instead that $u_1, \ldots, u_n$ are linear combinations of sufficiently smooth functions $v_1, \ldots, v_n$ having the property that $v_j(t) = t^{j-1} + o(t^n)$ as $t \to 0^+$, $j = 1, \ldots, n$.

3. Best Local Approximants and Consequences

As in Section 2, let S_n be the space spanned by $C^{n-1}[0, \delta]$ functions, $u_1, ..., u_n$, whose wronskian matrix evaluated at 0, denoted by W_n, is nonsingular. Let $f \in L_2[0, \delta]$ and $p_\epsilon(f)$ be the (unique) best $L_2[0, \epsilon]$ approximant of f from S_n. If f is smooth enough at 0, say of class $C^{n-1}[0, \delta]$, then the net $\{p_\epsilon(f)\}$, as $\epsilon \to 0^+$, has a unique cluster point, namely the best L_2 local approximant $p_0(f)$ of f. If we consider the function $f(t) = t^\alpha$, $n - 2 < \alpha < n - 1$, it is easy to show that the net $\{p_\epsilon(f)\}$, $0 < \epsilon \leqslant \delta$, of best $L_2[0, \epsilon]$ approximants from the space of all (algebraic) polynomials π_n with degree no greater than $n - 1$ has no cluster point. Hence, the condition $C^{n-1}[0, \delta]$ in Theorem 2.2 cannot be weakened to $C^\alpha[0, \delta]$ with $\alpha < n - 1$. However, the condition that $f \in C^{n-1}[0, \delta]$ is by no means necessary for the existence of best L_2 approximants. This can be seen from the example $f(x) = x \sin(1/x)$ and $S_n = \pi_2$; in this case by using the Riemann-Lebesgue Lemma, it is easy to show that $p_0(f) \equiv 0$.

For $f \in L_2[0, \delta]$, we let $P_0(f)$ be the set of all cluster points of the net $\{p_\epsilon(f)\}$, as $\epsilon \to 0^+$. We have seen examples where $P_0(f)$ is a singleton and where $P_0(f) = \varnothing$. We will next show that $P_0(f)$ may consist of a continuum of functions.

Example 3.1. *There exists an* $f^* \in C^\infty(0, 1] \cap C[0, 1]$ *such that* card $P_0(f^*) > 1$.

Proof. We first construct a discontinuous function f having the required property. Let $\{\rho_j\}$ be a positive sequence converging to 0 and set $\rho_0 = 1$. We only consider the case when $S_n = \pi_2$, the space of all linear polynomials. Let $P = P_{\pi_2}$ be the metric projection onto π_2. We define $f(t)$ to be $+t$ and $-t$ alternately so that $|f(t)| = t \leqslant 1$ on $[0, 1]$ as follows: First let $f(t) = t$ on $(\delta_1, \delta_0]$ where $\delta_0 = 1$ and $\delta_1 \in (0, 1)$ is chosen such that $\| P(f)(t) - t \|_{\delta_0} \leqslant \rho_0$. Even though f is not defined on $[0, \delta_1]$, the absolute continuity of the integral along with the fact that $|f| \leqslant 1$ assures that the above inequality makes sense. On $(\delta_2, \delta_1]$, we let $f(t) = -t$ where $\delta_2 \in (0, \delta_1)$ is chosen such that $\| P(f)(t) + t \|_{\delta_1} \leqslant \rho_1$. Using induction, we have a sequence $\delta_1 > \delta_2 > \cdots > 0$, $\delta_n \to 0$, such that $f(t) = (-1)^n t$ on $(\delta_{n-1}, \delta_n]$ and such that $\| P(f)(t) - (-1)^n t \|_{\delta_n} \leqslant \rho_n$. Hence, $P_0(f)$ contains the functions t and $-t$. By using standard smoothing techniques, we may change f to $f^* \in C^\infty(0, 1] \cap C(0, 1]$ so that $t, -t \in P_0(f^*)$.

We next show that under certain circumstances, $P_0(f)$ is a linear convex set.

Proposition 3.1. *Let* $f \in L_2[0, \delta]$, $\delta > 0$, *such that*

$$p_\epsilon(f) = \sum_{i=1}^{n} \alpha_{i,\epsilon} u_i$$

where $\alpha_{i,\epsilon} \to \alpha_i$, *as* $\epsilon \to 0^+$, *for* $i = 1,..., n-1$. *Then* $P_0(f)$ *is convex (a line segment or empty).*

Proof. Suppose that $P_0(f)$ is not empty and that $u, v \in P_0(f)$. Then we need only show that $(1-\lambda)u + \lambda v$ is in $P_0(f)$ for all λ, $0 < \lambda < 1$. Note that $u = \beta u_n + \sum_{j=1}^{n-1} \alpha_j u_j$ and $v = \gamma u_n + \sum_{j=1}^{n-1} \alpha_j u_j$. By definition, there exist $\eta_i \to 0^+$ and $\delta_i \to 0^+$ such that $p_{\eta_i}(f) \to u$ and $p_{\delta_i}(f) \to v$ as $i \to 0$. Since $p_\epsilon(f)$ is continuous as a function of ϵ (cf. (2.5)), by the Intermediate Value Theorem, for each $\lambda \in (0, 1)$ there is a sequence $\xi_i \to 0^+$ such that

$$p_{\xi_i}(f) = [(1-\lambda)\beta + \lambda\gamma] u_n + \sum_{j=1}^{n-1} \alpha_{j,\xi_i} u_j .$$

It follows that $p_{\xi_i}(f) \to (1-\lambda)u + \lambda v$, or $(1-\lambda)u + \lambda v \in P_0(f)$.

By applying Theorem 2.1 and the above result, we have the following

PROPOSITION 3.2. *Let* S_n *be the subspace of* $C^{n-1}[0, \delta]$, $\delta > 0$, *functions with a basis* $\{u_1, ..., u_n\}$ *such that* $\det W_n(u_1, ..., u_n; 0) \neq 0$. *Let* $f \in C^{n-2}[0, \delta]$. *Then the set* $P_0(f)$ *is either convex (a line segment) or empty.*

Next, we apply the results from Section 2 to quasi-rational approximation: Let P and Q be finite dimensional subspaces of continuous functions on $[0, \delta]$, $\delta > 0$. Let $m = \dim P$, $n = \dim Q$ and suppose that $\dim Q_0 = n_0 = n - 1$, where $Q_0 = \{q \in Q: q(0) = 0\}$. Let $Q_1 = \{q \in Q: q(0) = 1\}$. For an $f \in L_2[0, \delta]$, we consider the following minimization problem:

$$\inf\{\| fq - p \|_\epsilon : p \in P, q \in Q_1\}, \tag{3.1}$$

where $0 < \epsilon \leqslant \delta$. By a standard argument, it is easy to show that this problem always has a solution. If, in addition, the space

$$R_f = \{fq + p: p \in P, q \in Q_0\} \tag{3.2}$$

is a Haar subspace of continuous functions on $[0, \delta]$, it can also be shown that the solution $(p_\epsilon(f), q_\epsilon(f))$ is unique, and will be called the best $L_2[0, \epsilon]$ quasi-rational approximant pair of f from $P \times Q_1$.

Now, let $P, Q \subset C^{m+n-2}[0, \delta]$ where $m \geqslant 1$ and $n \geqslant 2$ and $\{p_1, ..., p_m\}$, $\{q_1, ..., q_{n_0}\}$ be bases of P and Q_0 respectively. For $f \in C^{m+n-2}[0, \delta]$, let $\phi_i = fq_{i-m}$ for $m < i \leqslant m + n_0$. Then $\{\phi_1, ..., \phi_{m+n_0}\}$ spans R_f. Assume that the wronskian matrix $W(f) \equiv W_{m+n_0}(\phi_1, ..., \phi_{m+n_0}; 0\}$ of $\{\phi_1, ..., \phi_{m+n_0}\}$ evaluated at 0 to be nonsingular. Then it follows that R_f is a Haar subspace with dimension $m + n_0$ (cf. [2]). Hence, f has a unique best $L_2[0, \epsilon]$ quasi-rational approximant pair $(p_\epsilon(f), q_\epsilon(f))$ from $P \times Q_1$ for every ϵ, $0 < \epsilon \leqslant \delta$. By a proof similar to that given in [2] and by using Theorem 2.2, we have the following

THEOREM 3.1. *Let* $f \in C^{m+n-2}[0, \delta]$, $\delta > 0$, *be such that* $\det W(f) \neq 0$, *and let* $(p_\epsilon(f), q_\epsilon(f))$, $0 < \epsilon \leqslant \delta$, *be the best* $L_2[0, \epsilon]$ *quasi-rational approximant of f from* $P \times Q_1$. *Then the net* $\{p_\epsilon(f), q_\epsilon(f)\}$, *as* $\epsilon \to 0^+$, *converges uniformly on some neighborhood of* 0 *to a pair* $(p_0(f), q_0(f)) \in P \times Q_1$. *Furthermore, for each* $j = 0, \ldots, m + n_0 - 1$,

$$(f q_0(f) - p_0(f))^{(j)}(0) = 0. \tag{3.3}$$

The equations in (3.3) are the Padé equations (cf. [1]). In [2], it is shown that if $f(x) = a_0 + \cdots + a_{m+n}x^{m+n} + 0(x^{m+n})$ is in $C^{m+n}[0, \delta]$, $\delta > 0$, then

$$\det W_{m+n+1}(1, \ldots, t^m, f(t)\,t, \ldots, f(t)\,t^n; 0) = \prod_{j=1}^{m} \prod_{i=1}^{n} j!\,(m+i)! \begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}$$

where $a_j \equiv 0$ for $j < 0$. Hence, we have the following

THEOREM 3.2. *Let* $P = \pi_{m+1}$, $Q = \pi_{n+1}$ *and* $f(x) = a_0 + \cdots + a_{m+n}x^{m+n} + o(x^{m+n}) \in C^{m+n}[0, \delta]$, $\delta > 0$, *such that*

$$\begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix} \neq 0$$

where $a_j \equiv 0$ *if* $j < 0$. *Let* (p_ϵ, q_ϵ), $0 < \epsilon \leqslant \delta$, *be the best* $L_2[0, \epsilon]$ *quasi-rational approximant of f from* $P \times Q_1$. *Then the net* $\{(p_\epsilon, q_\epsilon)\}$, *as* $\epsilon \to 0^+$, *converges uniformly on some neighborhood of* 0 *to the* $[m/n]$ *Padé approximant of f.*

4. FINAL REMARKS

In Proposition 3.1 we show, under rather restrictive hypotheses, that $P_0(f)$ is convex. It is not at all clear whether $P_0(f)$ is convex in general and in fact it is not even known whether $P_0(f)$ is connected. These questions seem to deserve further study. The analogous problems for L_p, $p \neq 2, \infty$, are still open.

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